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1974 J. Phys. A: Math. Nucl. Gen. 7 1323

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## Derivation of high field expansions for Ising model on the hydrogen peroxide lattice

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Received 27 November 1973

**Abstract.** The derivation of the low temperature (high field) series expansion for the free energy of the spin one-half Ising model on the three-dimensional coordination-three hydrogen peroxide lattice is discussed. Complete high field polynomials to order 23 are presented.

### 1. Introduction

This paper is concerned with the derivation of low temperature (high field) series expansions for the free energy of the spin one-half Ising model on a particular three-dimensional lattice, the hydrogen peroxide lattice. The general theory of low temperature series expansions for the Ising model has been presented in a recent series (Sykes *et al* 1965, to be referred to as I, Sykes *et al* 1973a, to be referred to as II). The third paper of the series (Sykes *et al* 1973b, to be referred to as III) deals with the high field expansion of the Ising model for two-dimensional lattices and in particular for the honeycomb lattice. The hydrogen peroxide lattice is the three-dimensional analogue of the honeycomb lattice, and much of the theory special to the honeycomb lattice in two dimensions applies equally to the hydrogen peroxide lattice in three dimensions.

Although there may not be any real magnetic insulator for which the magnetic ions lie on a hydrogen peroxide lattice, this lattice is of great theoretical interest. On the one hand, as is well known, the critical exponents are independent of lattice because in the critical region the range of correlations is much greater than the lattice spacing. On the other hand the present techniques for low temperature expansions are especially powerful on the hydrogen peroxide lattice. Finally it turns out for the hydrogen peroxide lattice (Betts and Chan 1974) as for the diamond lattice that the coefficients of the series for the thermodynamic properties along the coexistence curve are all of one sign.

As in previous papers in this series full details of the calculations will not be presented for the reason that very few workers will wish to repeat these or similar low temperature series calculations while the great majority of readers will be interested only in the final results. In § 2 we describe the rather unfamiliar hydrogen peroxide lattice and its shadow lattice, the hypertriangular lattice. In § 3 we outline the steps in the calculation needed to obtain the basic low temperature series. Section 4 contains a description of some checking procedures used to assure the correctness of the series coefficients. Section 5 compares

the series results for the three-dimensional hydrogen peroxide lattice with the infinite dimensional Bethe lattice. Finally the basic data, the partial generating functions  $F_1$  to  $F_{11}$  and the high field polynomials  $L_1(z)$  to  $L_{23}(z)$  for the hydrogen peroxide lattice are presented in the appendix.

The series for such thermodynamic properties as the specific heat, magnetization and susceptibility along the coexistence curve and along the critical isotherm can be derived readily from the  $L_s(z)$ . Such series and their analysis are presented in a companion paper (Betts and Chan 1974).

## 2. Hydrogen peroxide and hypertriangular lattices

Investigations of the Ising model and other models of cooperative phenomena in lattice statistics are usually carried out on regular lattices. A *regular* lattice is one in which all sites or vertices are equivalent and all bonds or edges are equivalent under a combination of translation and improper rotation. By restricting investigations to regular lattices the calculations are simplified without normally losing any of the interesting physics.

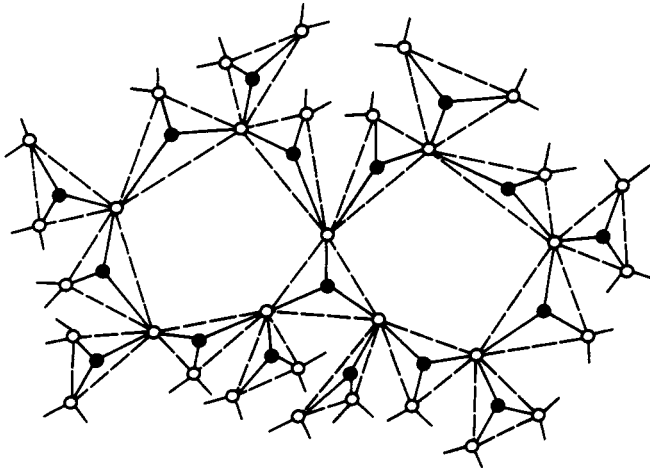
In two dimensions the only regular lattices are the honeycomb, square or quadratic, kagome and triangular lattices of coordination numbers,  $q = 3, 4, 4$  and  $6$  respectively. In three dimensions there are considerably more regular lattices all of which are necessarily of cubic symmetry. The three-dimensional lattices most often used in lattice statistical investigations are the diamond, simple cubic, body-centred cubic and face-centred cubic lattices with  $q = 4, 6, 8$  and  $12$  respectively.

In the present study we employ the hydrogen peroxide lattice discovered by Heesch and Laves (1933) and described by Wells (1954) as the only regular lattice with the theoretical minimum coordination number,  $q = 3$ . This lattice was introduced into lattice statistics by Leu *et al* (1969) who derived high temperature expansions for the Ising model partition function and susceptibility on the hydrogen peroxide lattice.

The hydrogen peroxide lattice consists of a BCC bravais lattice with four atoms per unit cell. The axes may be chosen so that the vertices of the hydrogen peroxide lattice are located at the points  $(l, m, n)a$ ,  $(l + \frac{1}{4}, m + \frac{1}{4}, n)a$ ,  $(l + \frac{1}{2}, m + \frac{1}{4}, n + \frac{1}{4})a$ ,  $(l + \frac{3}{4}, m, n + \frac{1}{4})a$ ,  $(l, m + \frac{3}{4}, n + \frac{3}{4})a$ ,  $(l + \frac{1}{4}, m + \frac{1}{2}, n + \frac{3}{4})a$ ,  $(l + \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2})a$  and  $(l + \frac{3}{4}, m + \frac{3}{4}, n + \frac{1}{2})a$ . Each vertex is surrounded by three nearest-neighbour vertices lying in a common plane and making  $120^\circ$  angles at the central vertex. In this respect the hydrogen peroxide lattice resembles the honeycomb lattice. It differs from the honeycomb lattice in that the plane of neighbours of one vertex is rotated by  $60^\circ$  with respect to the plane of neighbours of the neighbouring vertex. The smallest polygon which can be embedded in the hydrogen peroxide lattice has 10 edges and the second smallest has 14 edges.

In nature the only known examples of crystalline substances conforming to the hydrogen peroxide lattice are  $\text{H}_2\text{O}_2$  (oxygen atoms only) and  $\text{Hg}_3\text{S}_2\text{Cl}_3$  (Wells 1972). However as demonstrated in the succeeding sections the hydrogen peroxide lattice is of great theoretical interest in lattice statistics.

The hypertriangular lattice is derived from the hydrogen peroxide lattice as one of two equivalent sub-lattices of second-neighbour vertices. The hypertriangular lattice is to the hydrogen peroxide lattice as the triangular lattice is to the honeycomb lattice. The relationship is illustrated schematically in figure 1. This figure also illustrates that the hypertriangular lattice is the shadow lattice (Sykes *et al* 1965) to the hydrogen peroxide lattice. However these two models can be much better appreciated from a three-dimensional model.



**Figure 1.** Schematic relation between the hydrogen peroxide lattice, open circles and full lines, and the hypertriangular lattice, full circles and broken lines.

The infinite dimensional ( $d = \infty$ ) analogue of the honeycomb ( $d = 2$ ) and hydrogen peroxide ( $d = 3$ ) lattices is the  $q = 3$  Bethe lattice, which contains no closed circuits. The  $q = 3$  Bethe lattice consists of two equivalent sub-lattices of  $q = 6$ . The sub-lattice is a pure Husimi lattice consisting of three articulated triangles meeting at each vertex and is the  $d = \infty$  analogue of the triangular ( $d = 2$ ) and hypertriangular lattices ( $d = 3$ ).

### 3. Derivation of the complete low temperature codes $F_s$ for $s < 12$

The general theory of low temperature expansions for the spin one-half Ising model has been developed by Sykes *et al* (1965), and Sykes *et al* (1973a). Application to the high field expansion in two dimensions and in particular to the honeycomb–triangular pair of lattices has been expounded by Sykes *et al* (1973b). Most of the formalism for the honeycomb–triangular lattice ( $d = 2$ ) pair is equally applicable to the hydrogen peroxide–hypertriangular pair of lattices ( $d = 3$ ) or the  $q = 3$  Bethe and  $q = 6$  Husimi lattices ( $d = \infty$ ). Indeed in some ways the three-dimensional case is simpler because all triangles are of significant parity.

Using the theory as developed in I, II and III it is clear that all complete codes or partial generating functions on the hypertriangular lattice,  $F_n$  for  $n < 5$ , can be completely determined algebraically because of the smallest polygon on the hydrogen peroxide lattice being of ten sides. Similarly *all* complete codes on the Husimi lattice can be determined algebraically thus providing by an alternative method a complete solution of the Ising model on the Bethe lattice.

To complete  $F_5$  it is necessary to compute directly only the coefficient of the code (10, 5, 5, 0). This coefficient is the strong or low temperature lattice constant of the pentagon on the hypertriangular lattice and is readily found by inspection to be 3.  $F_5$  is then completed algebraically.  $F_6$  and  $F_7$  can also be readily completed by hand using theory of I, II and III. However, the graphical information required to compute  $F_8$  and beyond becomes rather extensive and further developments are required. For example

the coefficient of the code (16, 8, 8, 0) in  $F_8$  requires among others the lattice constant  $[\Sigma \text{ (graph) } ] = 1608$  and  $F_{11}$  requires  $[\Sigma \text{ (graph) } ] = 445146$ .

To obtain the complete codes  $F_s$  for  $s < 12$ , as we have done, we must :

- (i) determine all the codes  $(\lambda, \alpha, \beta, \gamma)$  whose coefficients must be supplied directly from configurational information rather than determined algebraically.
- (ii) List all the graphs corresponding to each code in (i).
- (iii) Compute the lattice constant of each graph in (ii).
- (iv) Solve the appropriate sets of linear equations to determine the coefficients of the remaining codes in  $F_s$ . We illustrate these steps for order  $s = 8$  (eight over turned spins) where  $3s = \alpha + 2\beta + 3\gamma$ .

Finally all needed lattice constants have been obtained by electronic computer using special programs written for the purpose by one of us (CJE).

The coefficients of the codes in  $F_s$  which can be obtained algebraically require at worst, for  $F_{11}$ , the simultaneous solution of eleven simultaneous equations in eleven unknowns in which the set of coefficients is triangular. Although it was easy to solve these equations on a desk calculator their solution was also computerized.

The resulting complete codes  $F_1$  to  $F_{11}$  are listed in the appendix. All coefficients are exact. The complete codes  $F_1$  to  $F_{11}$  derived for the hypertriangular lattice lead rather directly to the high field polynomials  $L_1$  to  $L_{23}$  on the hydrogen peroxide lattice where the logarithm of the partition function per site is

$$\ln \Lambda = \sum_s L_s(z)\mu^s. \tag{3.1}$$

These polynomials are also listed in the appendix.

Table I lists eight spin codes according to rank  $\gamma$ , (number of triangles) and energy,  $\alpha + \beta$  (power of  $u$ ). Codes above the broken line are non-graphical, that is, there are no graphs corresponding to these codes which are embeddable in the hypertriangular lattice. All eight spin codes of rank greater than three are non-graphical. Coefficients

**Table 1.** Eighth-order codes for the hydrogen peroxide-hypertriangular lattices.

				(10, 0, 6, 4)
				(11, 2, 5, 4)
			(11, 1, 7, 3)	(12, 4, 4, 4)
		(11, 0, 9, 2)	(12, 3, 6, 3)	(13, 6, 3, 4)
		(12, 2, 8, 2)	(13, 5, 5, 3)	(14, 8, 2, 4)
	(12, 1, 10, 1)	(13, 4, 7, 2)	(14, 7, 4, 3)	(15, 10, 1, 4)
(12, 0, 12, 0)	(13, 3, 9, 1)	(14, 6, 6, 2)	(15, 9, 3, 3)	(16, 12, 0, 4)
(13, 2, 11, 0)	(14, 5, 8, 1)	(15, 8, 5, 2)	(16, 11, 2, 3)	
(14, 4, 10, 0)	(15, 7, 7, 1)	(16, 10, 4, 2)	(17, 13, 1, 3)	
(15, 6, 9, 0)	(16, 9, 6, 1)	(17, 12, 3, 2)	(18, 15, 0, 3)	
(16, 8, 8, 0)	(17, 11, 5, 1)	(18, 14, 2, 2)		
(17, 10, 7, 0)	(18, 13, 4, 1)	(19, 16, 1, 2)		
(18, 11, 6, 0)	(19, 15, 3, 1)	(20, 18, 0, 2)		
(19, 12, 5, 0)	(20, 17, 2, 1)			
(20, 13, 4, 0)	(21, 19, 1, 1)			
(21, 14, 3, 0)	(22, 21, 0, 1)			
(22, 15, 2, 0)				
(23, 16, 1, 0)				
(24, 24, 0, 0)				

of codes below the solid line can be determined algebraically. Thus for  $s = 8$  the coefficients of only (15, 7, 7, 1), (15, 6, 9, 0) and (16, 8, 8, 0) need be calculated directly, while the remaining 24 graphical codes are calculated algebraically by exploiting the symmetry between the two sublattices.

The graphs which contribute to the required codes fall into two classes: (1) *basic star graphs* in which every vertex of every triangular subgraph is also a vertex of one or more polygons of higher degree; (2) other star graphs, trees and disconnected graphs. It is a straightforward matter to generate a list of class (2) graphs for each code of order  $s$  starting from a complete list of graphs of 'parent' codes of order  $s - 1$ . Graphs of class (1) are more difficult to obtain. We have obtained them from the list of stars on the hydrogen peroxide lattice as given by Leu *et al* (1969).

Low temperature lattice constants for all graphs have been obtained by direct inspection of the lattice as far as that is possible. Further lattice constants can be obtained from symbolic equations (Domb 1960).

#### 4. Checking procedures

All major steps in the derivation of the series as listed in the appendix have been computerized. In order to check the computer programs all stages of the calculation have also been carried out as far as is feasible by hand.

The coefficients of each group of codes of a given order  $s = (\alpha + 2\beta + 3\gamma)/3$  and rank  $\gamma$  forms a set among which there are  $s - \gamma$  linear relations when use is made of the sublattice symmetry and the complete lower codes are assumed known. Hence a complete check is afforded by 'overcounting', that is by determining the coefficient of one more code than necessary in each set. In practice it is easiest to obtain the coefficient of the code with largest  $\beta$  value. For  $F_8$  for example we have overcounted by finding coefficients of (15, 8, 5, 2), (16, 9, 6, 1) and (17, 10, 7, 0). All codes have been checked in this way with the exception of the codes for  $s = 11$ ,  $\gamma = 0$ .

A further check has been possible through another computer program which counts all *connected* configurations of  $s$  spins and sorts them by code. In this way the connected parts of the coefficients of all codes have been checked.

A further independent check is provided by the high-low transformation (Sykes *et al* 1966). The data from the *high temperature* series for the specific heat and susceptibility are used to provide two further constraints on each complete code. By this means we have also checked all  $F_s$  for  $s < 12$  and find agreement. Thus we believe our results for all  $F_s$  are correct.

#### 5. Comparison with Bethe lattice

The fact that the smallest polygon embeddable in the hydrogen peroxide lattice is of ten sides means that  $L_1$  to  $L_9$  for the hydrogen peroxide lattice are identical to  $L_1$  to  $L_9$  for the  $q = 3$  Bethe lattice. This raises the question as to what extent higher  $L_s$  differ between the two lattices, or to what extent the three dimensionality of the hydrogen peroxide lattice is sensed by the  $L_s$  for  $10 \leq s \leq 23$ .

We have computed the coefficients  $[s; r]_\infty$  in

$$L_s(z) = \sum_{r=0}^{s(s-1)/2} [s; r]_\infty z^{qs-2r} \quad (5.1)$$

for the  $q = 3$  Bethe lattice for comparison with the same coefficients,  $[s; r]_3$ , for the hydrogen peroxide lattice listed in the appendix.

The quantity  $p_{rs} = 1 - [s; r]_\infty / [s; r]_3$  is a convenient measure of how much the coefficients of the hydrogen peroxide lattice differ from the corresponding coefficients of the  $q = 3$  Bethe lattice. For  $s < 10$  of course all  $p_{rs} = 0$ . For  $s > 10$  the  $|p_{rs}|$  decrease monotonically with  $r$  for fixed  $s$  and the  $|p_{rs}|$  increase monotonically with  $s$  for fixed  $r$ . For  $s = 10$ ,  $p_{10,10} = 1$  but  $p_{12,10} = -0.003$  and for  $r_{\max} = 30$ ,  $p_{10,30} = 4 \times 10^{-6}$ . Thus  $L_{10}$  does not 'feel' the dimensionality of the lattice very much. For  $s = 16$ ,  $p_{14,16} = p_{16,16} = 1$  while  $p_{18,16} = -0.3$ . The  $|p_{r,16}|$  continue to decrease until for  $r_{\max} = 48$ ,  $p_{48,16} = 10^{-3}$ . For  $s = 23$ ,  $p_{17,23} = p_{19,23} = p_{21,23} = p_{23,23} = 1$ ,  $p_{25,23} = -5.1$  then  $|p_{r,23}|$  decline slowly through  $p_{37,23} = -0.1$  to  $p_{70,23} = -0.01$ . Thus the higher  $s$  polynomials are very much three-dimensional and we can usefully analyse the resulting series to predict thermodynamic properties of the three-dimensional Ising model on the critical isotherm.

The coefficients  $[r; s]$  can be regrouped to form  $\psi_r(\mu)$  in

$$\ln \Lambda = \sum_r \psi_r(\mu) z^r \quad (5.2)$$

where the present data give the first 17 complete  $\psi_r$ . Here the deviation from the Bethe lattice is more pronounced with  $p_{17,23} = p_{17,21} = p_{17,19} = p_{17,17} = 1$ ,  $p_{17,15} = -0.2$  with only  $p_{17,13}$  and  $p_{17,11}$  being very small. This means that series for thermodynamic functions on the coexistence curve will be still more truly three dimensional in their later coefficients than along the critical isotherm.

## Acknowledgments

Dr J A Leu was involved at an early stage of this investigation, Mr Marc Gryn timer helped with some of the configurational calculations and Dr J L Martin provided advice regarding computer programming. This research has been supported in part by the National Research Council of Canada.

## Appendix

Complete codes or partial generating functions  $F_s$  to order 11 and high field polynomials  $L_s(z)$  to order 23 for the spin half Ising model on the hydrogen peroxide lattice are listed here.

### Partial generating functions

$$F_1 = 1(3, 3, 0, 0)$$

$$F_2 = 3(5, 4, 1, 0) - 3\frac{1}{2}(6, 6, 0, 0)$$

$$F_3 = 1(7, 6, 0, 1) + 12(7, 5, 2, 0) - 33(8, 7, 1, 0) + 20\frac{1}{3}(9, 9, 0, 0)$$

$$F_4 = 12(9, 7, 1, 1) - 15(10, 9, 0, 1) + 56(9, 6, 3, 0) - 256\frac{1}{2}(10, 8, 2, 0) + 351(11, 10, 1, 0) \\ - 147\frac{3}{4}(12, 12, 0, 0)$$

$$F_5 = 3(11, 9, 0, 2) + 108(11, 8, 2, 1) - 297(12, 10, 1, 1) + 189(13, 12, 0, 1) + 3(10, 5, 5, 0) \\ + 273(11, 7, 4, 0) - 1862(12, 9, 3, 0) + 4164(13, 11, 2, 0) - 3798(14, 13, 1, 0) \\ + 1217\frac{1}{3}(15, 15, 0, 0)$$

$$F_6 = 66(13, 10, 1, 2) - 84\frac{1}{2}(14, 12, 0, 2) + 15(12, 7, 4, 1) + 820(13, 9, 3, 1) - 3810(14, 11, 2, 1) \\ + 5274(15, 13, 1, 1) - 2272(16, 15, 0, 1) + 30(12, 6, 6, 0) + 1329(13, 8, 5, 0) \\ - 12918(14, 10, 4, 0) + 41753(15, 12, 3, 0) - 61267\frac{1}{2}(16, 14, 2, 0) \\ + 41928(17, 16, 1, 0) - 10863\frac{1}{6}(18, 18, 0, 0)$$

$$F_7 = 13(15, 12, 0, 3) + 30(14, 9, 3, 2) + 846(15, 11, 2, 2) - 2475(16, 13, 1, 2) \\ + 1626(17, 15, 0, 2) + 2(13, 6, 6, 1) + 213(14, 8, 5, 1) + 5424(15, 10, 4, 1) \\ - 39370(16, 12, 3, 1) + 89130(17, 14, 2, 1) - 82398(18, 16, 1, 1) \\ + 26921(19, 18, 0, 1) + 237(14, 7, 7, 0) + 6321(15, 9, 6, 0) - 86502(16, 11, 5, 0) \\ + 378465(17, 13, 4, 0) - 789853(18, 15, 3, 0) + 859836(19, 17, 2, 0) \\ - 470943(20, 19, 1, 0) + 102477\frac{1}{7}(21, 21, 0, 0)$$

$$F_8 = 30(16, 11, 2, 3) + 360(17, 13, 1, 3) - 514(18, 15, 0, 3) + 12(15, 8, 5, 2) \\ + 585(16, 10, 4, 2) + 7830(17, 12, 3, 2) - 41880(18, 14, 2, 2) \\ + 60489(19, 16, 1, 2) - 26842\frac{1}{2}(20, 18, 0, 2) + 30(15, 7, 7, 1) \\ + 2107(16, 9, 6, 1) + 31620(17, 11, 5, 1) - 354405(18, 13, 4, 1) \\ + 1171774(19, 15, 3, 1) - 1739982(20, 17, 2, 1) + 1206828(21, 19, 1, 1) \\ - 318008(22, 21, 0, 1) + 12(15, 6, 9, 0) + 1653(16, 8, 8, 0) \\ + 28485(17, 10, 7, 0) - 561773(18, 12, 6, 0) + 3200181(19, 14, 5, 0) \\ - 8876036\frac{1}{4}(20, 16, 4, 0) + 13599827(21, 18, 3, 0) - 11751265\frac{1}{2}(22, 20, 2, 0) \\ + 5365728(23, 22, 1, 0) - 1006844\frac{7}{8}(24, 24, 0, 0)$$

$$F_9 = 15(18, 13, 1, 4) + 53(19, 15, 0, 4) + 30(17, 10, 4, 3) + 810(18, 12, 3, 3) \\ + 4980(19, 14, 2, 3) - 18537(20, 16, 1, 3) + 13093\frac{1}{3}(21, 18, 0, 3) + 3(16, 7, 7, 2) \\ + 237(17, 9, 6, 2) + 7194(18, 11, 5, 2) + 55605(19, 13, 4, 2) \\ - 530887(20, 15, 3, 2) + 1277937(21, 17, 2, 2) - 1220562(22, 19, 1, 2) \\ + 409342(23, 21, 0, 2) + 429(17, 8, 8, 1) + 17253(18, 10, 7, 1) \\ + 157754(19, 12, 6, 1) - 2887047(20, 14, 5, 1) + 13158120(21, 16, 4, 1) \\ - 27871169(22, 18, 3, 1) + 30690465(23, 20, 2, 1) - 17026500(24, 22, 1, 1) \\ + 3761584(25, 24, 0, 1) + 150(17, 7, 10, 0) + 10721(18, 9, 9, 0) \\ + 115080(19, 11, 8, 0) - 3542304(20, 13, 7, 0) + 25680324(21, 15, 6, 0) \\ - 90727959(22, 17, 5, 0) + 182833239(23, 19, 4, 0) - 220710559(24, 21, 3, 0) \\ + 157992594(25, 23, 2, 0) - 61856592(26, 25, 1, 0) + 10205103\frac{7}{9}(27, 27, 0, 0)$$



$$\begin{aligned}
F_{10} = & 3(20, 15, 0, 5) + 40(19, 12, 3, 4) + 600(20, 14, 2, 4) + 1332(21, 16, 1, 4) \\
& - 2870\frac{1}{2}(22, 18, 0, 4) + 21(18, 9, 6, 3) + 768(19, 11, 5, 3) + 12465(20, 13, 4, 3) \\
& + 41612(21, 15, 3, 3) - 364353(22, 17, 2, 3) + 583716(23, 19, 1, 3) \\
& - 273128(24, 21, 0, 3) + 54(18, 8, 8, 2) + 3453(19, 10, 7, 2) \\
& + 68791(20, 12, 6, 2) + 283014(21, 14, 5, 2) - 5562003(22, 16, 4, 2) \\
& + 20093143(23, 18, 3, 2) - 31022689\frac{1}{2}(24, 20, 2, 2) + 22094430(25, 22, 1, 2) \\
& - 5956953\frac{1}{2}(26, 24, 0, 2) + 30(18, 7, 10, 1) + 4553(19, 9, 9, 1) \\
& + 124833(20, 11, 8, 1) + 591822(21, 13, 7, 1) - 21695784(22, 15, 6, 1) \\
& + 132241344(23, 17, 5, 1) - 374687676(24, 19, 4, 1) + 581065166(25, 21, 3, 1) \\
& - 507614874(26, 23, 2, 1) + 234600657(27, 25, 1, 1) - 44632309(28, 27, 0, 1) \\
& + 2(18, 6, 12, 0) + 1401(19, 8, 11, 0) + 66247\frac{1}{2}(20, 10, 10, 0) \\
& + 353219(21, 12, 9, 0) - 21647458\frac{1}{2}(22, 14, 8, 0) + 197540979(23, 16, 7, 0) \\
& - 864268106(24, 18, 6, 0) + 2194249767\frac{3}{5}(25, 20, 5, 0) \\
& - 3459394701(26, 22, 4, 0) + 3440320179(27, 24, 3, 0) \\
& - 2101277061(28, 26, 2, 0) + 720068631(29, 28, 1, 0) \\
& - 106012306\frac{1}{5}(30, 30, 0, 0)
\end{aligned}$$

$$\begin{aligned}
F_{11} = & 30(21, 14, 2, 5) + 225(22, 16, 1, 5) + 99(23, 18, 0, 5) + 63(20, 11, 5, 4) \\
& + 1335(21, 13, 4, 4) + 11885(22, 15, 3, 4) + 10515(23, 17, 2, 4) \\
& - 113490(24, 19, 1, 4) + 94048(25, 21, 0, 4) + 6(19, 8, 8, 3) + 477(20, 10, 7, 3) \\
& + 12612(21, 12, 6, 3) + 138858(22, 14, 5, 3) + 139050(23, 16, 4, 3) \\
& - 5126311(24, 18, 3, 3) + 14279664(25, 20, 2, 3) - 14538162(26, 22, 1, 3) \\
& + 5080447(27, 24, 0, 3) + 936(20, 9, 9, 2) + 39078(21, 11, 8, 2) \\
& + 549948(22, 13, 7, 2) + 486944(23, 15, 6, 2) - 50422377(24, 17, 5, 2) \\
& + 261034239(25, 19, 4, 2) - 580425928(26, 21, 3, 2) + 657785232(27, 23, 2, 2) \\
& - 373158312(28, 25, 1, 2) + 84122153(29, 27, 0, 2) + 1(19, 6, 12, 1) \\
& + 540(20, 8, 11, 1) + 40755(21, 10, 10, 1) + 823013(22, 12, 9, 1) \\
& + 465462(23, 14, 8, 1) - 151588005(24, 16, 7, 1) + 1221415794(25, 18, 6, 1) \\
& - 4451485584(26, 20, 5, 1) + 9103288647(27, 22, 4, 1) \\
& - 11107281072(28, 24, 3, 1) + 8034192531(29, 26, 2, 1) \\
& - 3181420965(30, 28, 1, 1) + 531545985(31, 30, 0, 1) + 63(20, 7, 13, 0) \\
& + 11434(21, 9, 12, 0) + 391650(22, 11, 11, 0) + 66906(23, 13, 10, 0) \\
& - 127557437(24, 15, 9, 0) + 1465184946(25, 17, 8, 0) \\
& - 7788846597(26, 19, 7, 0) + 24192018158(27, 21, 6, 0)
\end{aligned}$$

$$\begin{aligned}
 & -47718194586(28, 23, 5, 0) + 61656579231(29, 25, 4, 0) \\
 & -52088959132(30, 27, 3, 0) + 27736974918(31, 29, 2, 0) \\
 & -8451120537(32, 31, 1, 0) + 1123450617\frac{1}{11}(33, 33, 0, 0)
 \end{aligned}$$

High field polynomials

$$L_1 = z^3$$

$$L_2 = 1\frac{1}{2}z^4 - 2z^6$$

$$L_3 = 3z^5 - 9z^7 + 6\frac{1}{3}z^9$$

$$L_4 = 7z^6 - 33\frac{3}{4}z^8 + 51z^{10} - 24\frac{1}{2}z^{12}$$

$$L_5 = 18z^7 - 121z^9 + 288z^{11} - 291z^{13} + 106\frac{1}{3}z^{15}$$

$$L_6 = 49\frac{1}{2}z^8 - 429z^{10} + 1410\frac{1}{2}z^{12} - 2220z^{14} + 1684\frac{1}{2}z^{16} - 495\frac{3}{3}z^{18}$$

$$L_7 = 143z^9 - 1521z^{11} + 6420z^{13} - 13872z^{15} + 16281z^{17} - 9888z^{19} + 2437\frac{1}{7}z^{21}$$

$$\begin{aligned}
 L_8 = & 429z^{10} - 5414\frac{1}{2}z^{12} + 27999z^{14} - 77398\frac{7}{8}z^{16} + 124165z^{18} - 116077\frac{1}{2}z^{20} \\
 & + 58755z^{22} - 12457\frac{1}{4}z^{24}
 \end{aligned}$$

$$\begin{aligned}
 L_9 = & 1326z^{11} - 19380z^{13} + 118864z^{15} - 401793z^{17} + 822360z^{19} - 1047420z^{21} \\
 & + 813216z^{23} - 352791z^{25} + 65618\frac{1}{3}z^{27}
 \end{aligned}$$

$$\begin{aligned}
 L_{10} = & 1\frac{1}{2}z^{10} + 4184z^{12} - 69700\frac{1}{2}z^{14} + 495463\frac{1}{2}z^{16} - 1984976z^{18} + 4959838\frac{4}{3}z^{20} \\
 & - 8042325z^{22} + 8485862\frac{1}{2}z^{24} - 5631619\frac{1}{2}z^{26} + 2137315\frac{1}{2}z^{28} \\
 & - 354044\frac{9}{10}z^{30}
 \end{aligned}$$

$$\begin{aligned}
 L_{11} = & 15z^{11} + 13371z^{13} - 251461z^{15} + 2037402z^{17} - 9458250z^{19} + 27985130z^{21} \\
 & - 55320660z^{23} + 74182779z^{25} - 66776053z^{27} + 38687790z^{29} \\
 & - 13048224z^{31} + 1948161\frac{1}{11}z^{33}
 \end{aligned}$$

$$\begin{aligned}
 L_{12} = & 97\frac{1}{2}z^{12} + 43029z^{14} - 908463\frac{3}{4}z^{16} + 8286908\frac{1}{3}z^{18} - 43833172\frac{1}{2}z^{20} + 150201585z^{22} \\
 & - 351274376\frac{1}{4}z^{24} + 573569217z^{26} - 655220490z^{28} + 514088399z^{30} \\
 & - 264244883\frac{1}{4}z^{32} + 80188977z^{34} - 10896827\frac{1}{6}z^{36}
 \end{aligned}$$

$$\begin{aligned}
 L_{13} = & 525z^{13} + 138830z^{15} - 3281880z^{17} + 33392535z^{19} - 198664785z^{21} + 775265400z^{23} \\
 & - 2098448703z^{25} + 4050218774z^{27} - 5627965959z^{29} + 5597194944z^{31} \\
 & - 3891121690z^{33} + 1797133827z^{35} - 495661896z^{37} + 61800078\frac{1}{13}z^{39}
 \end{aligned}$$

$$\begin{aligned}
 L_{14} = & z^{12} + 2536\frac{1}{2}z^{14} + 447481\frac{1}{2}z^{16} - 11841776z^{18} + 133444096\frac{1}{2}z^{20} - 883929669z^{22} \\
 & + 3877211902z^{24} - 11944978402\frac{1}{2}z^{26} + 26666016898\frac{3}{7}z^{28} \\
 & - 43765361029\frac{1}{2}z^{30} + 52896457630\frac{1}{2}z^{32} - 46565296611z^{34} \\
 & + 29055837352z^{36} - 12182650104z^{38} + 3079314606z^{40} \\
 & - 354674912\frac{1}{14}z^{42}
 \end{aligned}$$

$$\begin{aligned}
L_{15} = & 12z^{13} + 11425z^{15} + 1436253z^{17} - 42636642z^{19} + 529247008z^{21} - 3871587153z^{23} \\
& + 18889738137\frac{3}{5}z^{25} - 65375406444\frac{1}{3}z^{27} + 166021444314z^{29} \\
& - 314961411369z^{31} + 449306269607z^{33} - 480296343859\frac{4}{3}z^{35} \\
& + 379171416132z^{37} - 214589972615z^{39} + 82377179820z^{41} \\
& - 19215911169z^{43} + 2056526543\frac{3}{5}z^{45}
\end{aligned}$$

$$\begin{aligned}
L_{16} = & 93z^{14} + 49018\frac{1}{2}z^{16} + 4575139z^{18} - 153062158\frac{1}{2}z^{20} + 2084270379z^{22} \\
& - 16727473621\frac{1}{4}z^{24} + 90014451033z^{26} - 346294592671\frac{1}{2}z^{28} \\
& + 987391158963z^{30} - 2129744151758\frac{7}{16}z^{32} + 3509772030531z^{34} \\
& - 4425412933290\frac{1}{2}z^{36} + 4239407973591z^{38} - 3033241322682\frac{3}{4}z^{40} \\
& + 1570453365285z^{42} - 555911608111\frac{1}{2}z^{44} + 120389303730z^{46} \\
& - 12032033468\frac{1}{8}z^{48}
\end{aligned}$$

$$\begin{aligned}
L_{17} = & 598z^{15} + 202764z^{17} + 14409894z^{19} - 547450096z^{21} + 8153604795z^{23} \\
& - 71405101536z^{25} + 420833846469z^{27} - 1784180286798z^{29} \\
& + 5651844105765z^{31} - 13680739580495z^{33} + 25620479515527z^{35} \\
& - 37298644730412z^{37} + 42120959081893z^{39} - 36547100200164z^{41} \\
& + 23906230325115z^{43} - 11405426399510z^{45} + 3745493261637z^{47} \\
& - 756919374921z^{49} + 70954769475\frac{1}{17}z^{51}
\end{aligned}$$

$$\begin{aligned}
L_{18} = & 1\frac{1}{2}z^{14} + 3433\frac{1}{2}z^{16} + 814269z^{18} + 44669941\frac{1}{2}z^{20} - 1949392143z^{22} + 31692763208z^{24} \\
& - 301524958312\frac{1}{2}z^{26} + 1934917700613z^{28} - 8975633689717\frac{1}{2}z^{30} \\
& + 31314976037650\frac{1}{2}z^{32} - 84172872303429z^{34} + 176820821627237\frac{2}{3}z^{36} \\
& - 292389067189908z^{38} + 381048704624616z^{40} - 389545709247428\frac{1}{2}z^{42} \\
& + 308834552509341z^{44} - 186042063589293z^{46} + 82293071837496z^{48} \\
& - 25202642440527z^{50} + 4774062085647z^{52} - 421381862696\frac{2}{3}z^{54}
\end{aligned}$$

$$\begin{aligned}
L_{19} = & 21z^{15} + 18189z^{17} + 3188091z^{19} + 135471942z^{21} - 6905938272z^{23} \\
& + 122423195703z^{25} - 1260778496526z^{27} + 8765897452023z^{29} \\
& - 44222743718502z^{31} + 168700384554780z^{33} - 499182804413709z^{35} \\
& + 1163951275197360z^{37} - 2158054973336719z^{39} + 3193111724081010z^{41} \\
& - 3765364402180404z^{43} + 3515509230247087z^{45} - 2565449575498857z^{47} \\
& + 1432096755422085z^{49} - 590437841068988z^{51} + 169401214193559z^{53} \\
& - 30197254445136z^{55} + 2518236075263\frac{1}{19}z^{57}
\end{aligned}$$

$$\begin{aligned}
L_{20} = & 184\frac{1}{2}z^{16} + 90651\frac{1}{2}z^{18} + 12201501\frac{3}{4}z^{20} + 398442687z^{22} - 24322143193\frac{1}{2}z^{24} \\
& + 470014161397\frac{1}{2}z^{26} - 5224181339661\frac{3}{4}z^{28} + 39190590764771\frac{2}{5}z^{30} \\
& - 213915411067809\frac{3}{4}z^{32} + 886804880651334z^{34} - 2867679902319284z^{36}
\end{aligned}$$

$$\begin{aligned}
 &+ 7357764343318974z^{38} - 15136198582450644z^{40} \\
 &+ 25101400467745347z^{42} - 33592891698515892z^{44} \\
 &+ 36162670173938031z^{46} - 31064085272018125\frac{3}{4}z^{48} \\
 &+ 20997085086857609\frac{7}{10}z^{50} - 10919743335146958\frac{3}{4}z^{52} \\
 &+ 4215633039979898z^{54} - 1137622611924962\frac{1}{4}z^{56} \\
 &+ 191500766930638\frac{1}{2}z^{58} - 15134458156494\frac{3}{20}z^{60}
 \end{aligned}$$

$$\begin{aligned}
 L_{21} = &1323z^{17} + 430182z^{19} + 45718229z^{21} + 1120539792z^{23} - 85095886233z^{25} \\
 &+ 1793619480582\frac{2}{3}z^{27} - 21465431325378z^{29} + 173127804877254z^{31} \\
 &- 1017935276525102z^{33} + 4561835414102031\frac{3}{2}z^{35} \\
 &- 16021945036844910z^{37} + 44905908255854368z^{39} \\
 &- 101613354282959886z^{41} + 186903731997443451z^{43} \\
 &- 280245540313157249z^{45} + 342251744594589351z^{47} \\
 &- 338849726998604449\frac{2}{7}z^{49} + 269498195865459620z^{51} \\
 &- 169636964749506429z^{53} + 82572772040916168z^{55} \\
 &- 29970234585965632z^{57} + 7634012726533563z^{59} \\
 &- 1217293609514970z^{61} + 91421894344323\frac{5}{21}z^{63}
 \end{aligned}$$

$$\begin{aligned}
 L_{22} = &3z^{16} + 8370\frac{1}{2}z^{18} + 1959481\frac{1}{2}z^{20} + 167840194\frac{1}{2}z^{22} + 2935273028z^{24} \\
 &- 295519270846\frac{1}{2}z^{26} + 6803493293622z^{28} - 87504541802766\frac{1}{2}z^{30} \\
 &+ 756493598609616z^{32} - 4773041146564965z^{34} + 23018561402772517z^{36} \\
 &- 87341709282688629z^{38} + 265760126833423944z^{40} \\
 &- 656672693675081495\frac{1}{2}z^{42} + 1328140280244409195\frac{7}{11}z^{44} \\
 &- 2208031854541089061\frac{1}{2}z^{46} + 3020206623133842152\frac{1}{2}z^{48} \\
 &- 3391105526002817485\frac{1}{2}z^{50} + 3107406848381889535\frac{1}{2}z^{52} \\
 &- 2300652130584331214\frac{1}{2}z^{54} + 1354920776966205385\frac{1}{2}z^{56} \\
 &- 619818322654171266z^{58} + 212262173815534014\frac{1}{2}z^{60} \\
 &- 51195290584927827z^{62} + 7754484620018985z^{64} \\
 &- 554807062334488\frac{2}{11}z^{66}
 \end{aligned}$$

$$\begin{aligned}
 L_{23} = &48z^{17} + 48441z^{19} + 8617406z^{21} + 603774576z^{23} + 6747317997z^{25} \\
 &- 1017723279317z^{27} + 25651310436624z^{29} - 354061536754941z^{31} \\
 &+ 3272477752646354z^{33} - 22083326312643045z^{35} \\
 &+ 114155720998785930z^{37} - 465813704970417072z^{39} \\
 &+ 1530538682933110038z^{41} - 4104128077094857968z^{43} \\
 &+ 9061167056974367956z^{45} - 16559112659094354042z^{47}
 \end{aligned}$$

$$\begin{aligned}
&+ 25105803765794858595z^{49} - 31562148720832543073z^{51} \\
&+ 32788899493478101590z^{53} - 27960461548415023320z^{55} \\
&+ 19361912137372503092z^{57} - 10712589275796094284z^{59} \\
&+ 4622167746869248128z^{61} - 1498274806850039777z^{63} \\
&+ 343138747923146652z^{65} - 49495440902354931z^{67} \\
&+ 3381150761398343\frac{1}{23}z^{69}
\end{aligned}$$

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